

# Spin liquid mediated RKKY interaction

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We propose an RKKY-type interaction that is mediated by a spin liquid. If a spin liquid ground state exists such an interaction could leave a fingerprint by ordering underlying localized moments such as nuclear spins. This interaction has a unique phenomenology that is distinct from the RKKY interaction found in fermionic systems; most notably the lack of a Fermi surface and absence of the requirement for itinerant electrons, since most spin liquids are insulators. As a working example we investigate the two-dimensional spin-1/2 kagome antiferromagnet (KAFM), although the treatment remains general and can be extended to other spin liquids and dimensions. We find that several different nuclear spin orderings minimize the RKKY-type energy induced by the KAFM but are unstable due to a zero-energy flat magnon band. Despite this we show that a small magnetic field is able to gap out this magnon spectrum for some of the orderings resulting in an intricate nuclear magnetism.

## I. INTRODUCTION

Quantum spin liquids are highly correlated spin systems where magnetic order is prevented due to frustration [1, 2]; that is to say that competing forces prevent all interactions from being simultaneously minimized. Spin liquids are characterized by no magnetic ordering occurring at any finite temperature and can be distinguished by their high degree of long-ranged entanglement. In this paper we show how the dynamics of spin liquid can mediate a long-range interaction that can lead to unique signatures of the underlying spin liquid state. Such an interaction is akin to the Ruderman-Kittel-Kasuya-Yosida (RKKY) interaction found in fermionic systems [3–5]. For instance a local spin liquid excitation can be produced through an exchange coupling to a different species of magnetic moments. Due to the strong spin liquid correlations this excitation can spread far over the system before coupling to another magnetic moment of the different species elsewhere in the material and, as a result, producing an effective, long-ranged Heisenberg type interaction between these magnetic moments. This is the same mechanism that is behind the usual RKKY interaction, although basic features such as the characteristic  $2k_F$  oscillations (with  $k_F$  the Fermi momentum) will be absent in the spin liquid case and we can expect distinct consequences. In this work we shall explicitly calculate this spin-liquid-RKKY (slRKKY) interaction and investigate its impact on magnetic ordering of localized magnetic moments embedded in a spin liquid.

As a working example we will focus on a two dimensional kagome spin liquid coupled to nuclear spins, as shown in Fig. 1; although our treatment will remain general and will also be applicable to different spin liquids - of any dimensions - and to other types of localized moments. Nuclear spins are, however, naturally embedded in the spin liquid on the ions of the crystalline lattice and so do not need to be added by further material engineering, as well as being accessible through the magnetic resonance techniques of NMR. The hyperfine interaction

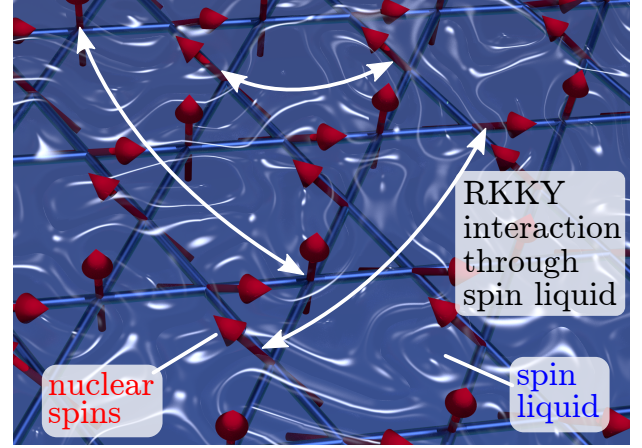


FIG. 1. Nuclear spins (red arrows) embedded in a two-dimensional spin liquid (symbolized by the blue liquid) on the kagome lattice interact through slRKKY, an RKKY like effective interaction, (illustrated by the selection of white arrows) that is mediated through the correlated dynamics of the spin liquid. In the presence of a magnetic field this interaction can lead to a nuclear magnetic order as shown by the red arrows.

provides the coupling between the two spin species.

In two dimensions a prime candidate for a real material that holds a quantum spin liquid state is the  $S = 1/2$  Heisenberg antiferromagnet on the kagome lattice [6–8]. Another possibility is the Kitaev model on the honeycomb lattice [9], although the simplicity of kagome’s geometric frustration makes it a more tractable working example our analysis should be equally applicable to Kitaev materials. We will focus on the two dimensional case however we expect many of the general features of slRKKY found within this work are equally applicable to three dimensional materials that possess a spin liquid ground state, such as the hyper-kagome lattice [10] or the

3D Kitaev materials [11].

The kagome antiferromagnet (KAFM) is made up of corner-sharing equilateral triangles with each site shared by only two elementary triangles (see Figs. 1 & 2). As such the kagome lattice takes advantage of the inherent geometric frustration of the triangular motif but has a reduced number of ground state constraints compared to the triangular lattice. This has the effect of increasing the size of the ground state degeneracy of kagome compared to that of the triangular lattice. It is for this reason that the  $S = 1/2$  nearest-neighbor Heisenberg model on the kagome lattice is now known to contain a quantum spin liquid state at low temperatures, whereas the  $S = 1/2$  triangular lattice orders [6].

In the last decade several experimental KAFM candidates have been studied [8, 12, 13]. Two notable examples are the ‘structurally perfect’  $\text{ZnCu}_3(\text{OH})_6\text{Cl}_2$  (Herbertsmithite) [14] and  $[\text{NH}_4]_2[\text{C}_7\text{H}_{14}\text{N}][\text{V}_7\text{O}_6\text{F}_{18}]$  (DQVOF) [15]. The suitability of these two candidates to hold a quantum spin liquid ground state is exemplified by the fact that no freezing of electron spins has been observed in Herbertsmithite or DQVOF down to temperatures of 50 and 40mK respectively [16, 17].

In such a system the nuclear spins form a Kondo type lattice in which the spin liquid state takes the role of both the mediator of sLRKKY interaction and of Kondo type screening. Although a variant of Kondo lattice physics may be possible or a variant of general Kondo physics in a spin liquid may be possible [18–26] our focus will be different because in most cases the nuclear spins are much larger than  $1/2$  and as a result the semi-classical treatment below becomes accurate. To understand the full implications of the sLRKKY interactions we will consider the stabilization of nuclear magnetic order as any ordering of the nuclear spins could affect measurements of the spin liquid.

When considering the possibility of the appearance of long-range order in a low-dimensional system it is important to make the distinction between the possibility of order appearing in principle and the possibility of it appearing in practice. Whether the appearance of order can occur in principle is addressed for a large class of systems by the Mermin-Wagner theorem [27]. The theorem forbids long-range magnetic order in Heisenberg type systems when they satisfy the condition that the interaction is sufficiently short ranged (decaying faster than  $1/r^{2+d}$  with  $r$  the distance and  $d$  the dimensionality of the system). This condition is not satisfied for the system under consideration here and so the Mermin-Wagner theorem is not applicable.

That said, the requirement for short range interactions has been partially lifted by an extension of the theorem by Loss, Pedrocchi, and Leggett [28], who rigorously proved the absence of long-range order in low dimensions for RKKY systems with arbitrary electron-electron interactions, provided that the interactions are isotropic and the RKKY interaction is carried by itinerant electrons. Since the majority of spin liquids systems have localized

electron spins this extended Mermin-Wagner theorem is also not applicable here.

However, since the theorem is invariant even when interactions are included it is likely that a further extension of the theorem will hold for sLRKKY. In this work we show via explicit calculation that, for the considered sLRKKY system, it is indeed the case that the order breaks down if we maintain isotropy in nuclear spin space but that we can stabilize this order by breaking that isotropy with a small magnetic field.

Yet the exclusion of long-range order *in principle* must further be contrasted with long-range order *in practice*. If the order destabilizing fluctuations become effective only at wavelengths larger than the sample size then order can extend over the entire sample and remain stable up to reasonable, accessible temperatures. For regular RKKY systems strong correlations induced by electron-electron interactions can play a crucial stabilizing role here, as was demonstrated for two-dimensional [29, 30] and one-dimensional conductors [31–34]. The generality of the ordering mechanism has been further exploited to find self-sustained topological phases when the electrons are driven into a superconducting state [35–37]. As most of this physics depends on the existence of an electronic Fermi surface, i.e. on  $k_F$ , a distinct phenomenology exists for sLRKKY.

One notable consequence resulting from the lack of a Fermi-surface  $k_F$  is that the minimum wavelength for destabilizing fluctuations is set by the Brillouin zone boundaries and so even finite systems will have very low ordering temperatures unless they are extremely small - at which sizes they may no longer exhibit spin liquid behaviour. Luckily this means that such an interaction would not invalidate the conclusions of experiments which probe whether real materials are spin liquids. However the unique nature of sLRKKY also enables a unique solution to the problem of ordering: The nuclear moments form a lattice which can result in intricate orderings, including those with a net magnetic moment. By applying a magnetic field we can couple to the magnon spectrum of the nuclear spins ordered with a net magnetic moment and induce a gap, thereby ordering the nuclear spins below a temperature that scales with the nuclear Zeeman energy and is therefore potentially accessible experimentally.

The main results of this work are as follows: We show that an RKKY-type interaction can be mediated by the electron spins that form a spin liquid. We use the  $S = 1/2$  kagome antiferromagnet (KAFM) as a working example but our treatment remains generally applicable to other spin liquids. The exchange interaction of sLRKKY is governed by the static magnetic susceptibility of the KAFM which we calculate using a second order Kondo-Yamaji Green’s function decoupling, although our results are independent of the method used to calculate the susceptibility. We perform an extended Luttinger-Tisza-type mean-field calculation to find the ordering vectors where the minimum energy of the sLRKKY Hamiltonian

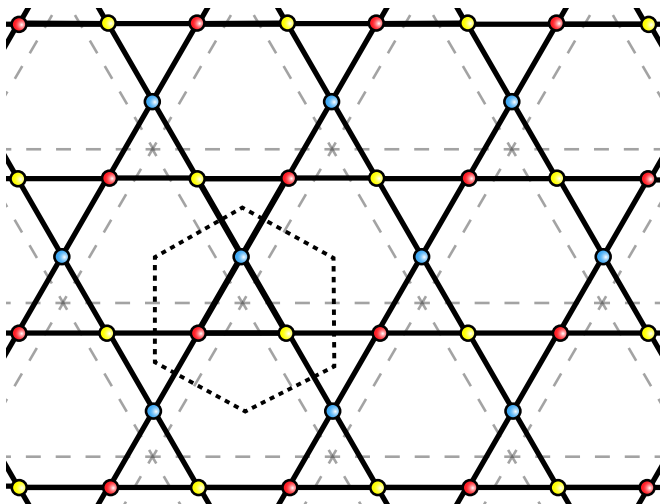


FIG. 2. The kagome lattice, a tripartite lattice with three sites per unit cell (here the unit cell is shown by the black dashed lines and sublattices by red, blue, and yellow points). It is built upon the triangular Bravais lattice (here shown by gray dashed lines).

can be achieved. Subsequently we demonstrate that these minima are unstable against magnon fluctuations due to the existence of a flat magnon band, regardless of whether the system is infinite or finite. Finally we show that for the specific case of the KAFM an ordering with a net magnetic moment can be stabilized, at potentially experimentally obtainable temperatures, by applying a small magnetic field.

## II. MODELING THE EFFECTIVE RKKY INTERACTION ON KAGOME

The kagome lattice (shown in Fig. 2) is a tripartite lattice built upon the triangular Bravais lattice. It takes advantage of the inherent geometric frustration of a triangular motif and, as a result, the spin-1/2 KAFM holds a spin liquid state at low temperature [1]. The straightforward nature of the exchange interaction means that the KAFM is the simplest and best working example for sRKKY in 2D.

We assume that the Hamiltonian of the whole system - spin liquid and its interaction with the nuclear spins - can be written in the form [32, 35]

$$H = H_{\text{sl}} + H_{\text{hyp}}; \quad (1)$$

where  $H_{\text{hyp}}$  is the Hamiltonian of the hyperfine interaction between the electron and nuclear spins,

$$H_{\text{hyp}} = A \sum_i \mathbf{S}_i \cdot \mathbf{I}_i, \quad (2)$$

with the strength of the the interaction,  $A$ , either positive or negative;  $H_{\text{sl}}$  is the Hamiltonian of the spin liquid, in

our case describing a nearest-neighbor Heisenberg model on the KAFM [38]

$$H_{\text{sl}} = \frac{1}{2} \sum_{i,j} J_{ij} \mathbf{S}_i \cdot \mathbf{S}_j, \quad (3)$$

such that

$$J_{ij} = \begin{cases} J > 0, & \text{if } i \text{ and } j \text{ are nearest neighbors} \\ 0, & \text{otherwise;} \end{cases} \quad (4)$$

and  $\mathbf{S}_i = (S_i^x, S_i^y, S_i^z)$  and  $\mathbf{I}_i = (I_i^x, I_i^y, I_i^z)$  are the spin operators at lattice site  $i$  of the localized electron and nuclear spins respectively. We consider nuclear spins located on the same site as the electron spin and assume that the nuclear moments are large and can be treated semi-classically. For the known  $S = 1/2$  KAFM candidates this is a reasonable assumption, for example in Herbertsmithite the equivalent moments correspond to the  $I = 3/2$  copper ions and in DQVOF the  $I = 7/2$  vanadium ions [39]. Away from the KAFM the  $I = 3/2$  iridium nuclear spins of the Kitaev iridates [40] show the more general applicability of our treatment to other potential spin liquid candidates.

We also make the normal RKKY assumption that  $|A|/J \ll 1$ , this is reasonable since in most materials  $|A|/k_B$  is of the order of 10mK whereas  $J/k_B$  is of the order of 100K, for example  $J/k_B \sim 60\text{K}$  is found in DQVOF [17] (with  $k_B$  the Boltzmann's constant). In this regime the electron spin relaxation times are much faster than typical time-scales of nuclear spin relaxation. The result is that we can decouple the electron and nuclear spin systems with the effective spin-spin interaction of the nuclear spins carried by the magnetic susceptibility of the electron spins. As such we can perform the standard transformation to an effective RKKY Hamiltonian of the nuclear spins such that (see Appendix A for details)

$$H_{\text{eff}} = \frac{A^2}{4J} \sum_{\mathbf{q}, a, b} \chi^{ab}(\mathbf{q}, \omega = 0) \mathbf{I}_{\mathbf{q}}^a \cdot \mathbf{I}_{-\mathbf{q}}^b, \quad (5)$$

where  $a, b$  refer to the lattice sites of the kagome unit cell (sublattice, cf. Fig 2);

$$\mathbf{I}_{\mathbf{q}}^a = \frac{1}{\sqrt{N_{Br}}} \sum_n \mathbf{I}_n^a e^{-i\mathbf{R}_n \cdot \mathbf{q}} \quad (6)$$

is the Fourier transform of the nuclear spin operators on sublattice  $a$  with respect to the sites  $\mathbf{R}_n$  of the underlying triangular Bravais lattice;  $N_{Br}$  the number of sites on the underlying triangular Bravais lattice (see Fig. 2); and

$$\chi^{ab}(\mathbf{q}, 0) = -iJ \int_0^\infty dt \left\langle \left[ S_{\mathbf{q}}^{a,(+)}(t), S_{-\mathbf{q}}^{b,(-)}(0) \right] \right\rangle e^{-\eta t} \quad (7)$$

is the static electron spin susceptibility with  $\eta \rightarrow 0^+$  (and we have set  $\hbar = 1$ ). We use the susceptibility of

$S^+$ ,  $S^-$  because correlations in the non-magnetic spin liquid phase of the electron spins are isotropic [38] such that  $\langle S_i^+ S_j^- \rangle = 2\langle S_i^z S_j^z \rangle$ .

There is the possibility that after ordering there will exist a reciprocal action of the nuclear moments on the spin liquid with the nuclear spins forming an effective magnetic field (Overhauser field) that has a destabilising effect on the spin liquid. In the present work we shall not consider this scenario since the strength of the field will be small  $A\langle I \rangle \ll J$  and so will act as a negligible perturbation to the Heisenberg Hamiltonian of the electron spins.

### III. THE SPIN SUSCEPTIBILITY OF THE $S = 1/2$ KAFM

A second-order Green's function decoupling introduced by Kondo and Yamaji [41] can be used to calculate the spin susceptibility of the  $S = 1/2$  KAFM, this has previously been utilised to calculate the specific heat and structure factor of the KAFM [38, 42] and we follow those previous calculation closely in this section. The positions in reciprocal space where the RKKY energy can be minimized will only depend upon certain qualitative features of the susceptibility, namely the positions of degenerate eigenvalues, as such the orderings found in Sec. V are independent of the method used to calculate the susceptibility.

A first order decoupling of the spin Green's function - commonly known as a Tyablikov decoupling [43] - can be achieved by expanding in terms of expectation values of spin operators. Beginning with the equation of motion for the dynamic spin susceptibility

$$\begin{aligned} \omega \chi_{ij}(\omega) &= \omega \langle \langle S_i^+; S_j^- \rangle \rangle_\omega \\ &= \langle \langle S_i^+, S_j^- \rangle \rangle_{\omega=0} + \langle \langle [S_i^+, H]; S_j^- \rangle \rangle_\omega \\ &= 2\delta_{ij} \langle S_i^z \rangle_0 + \sum_k J_{ik} \langle \langle S_i^z S_k^+ - S_i^+ S_k^z; S_j^- \rangle \rangle_\omega, \end{aligned} \quad (8)$$

where in the second line we have inserted the Heisenberg Hamiltonian (3) for the electron spins and  $\langle \langle A; B \rangle \rangle_\omega$  is the time Fourier transform of the retarded Green's function of  $A$  and  $B$ ,

$$\langle \langle A; B \rangle \rangle_t = -i\theta(t) \langle [A(t), B(0)] \rangle. \quad (9)$$

In a spin liquid there is no long-ranged order and as such  $\langle S_i^x \rangle_0 = \langle S_i^y \rangle_0 = \langle S_i^z \rangle_0 = 0$  this means that a standard first order Tyablikov decoupling of the form

$$\langle \langle S_i^z S_k^+; S_j^- \rangle \rangle_\omega \rightarrow \langle S_i^z \rangle_0 \langle \langle S_k^+; S_j^- \rangle \rangle_\omega, \quad (10)$$

will not allow us to calculate the spin-susceptibility from a self-consistent identity. Instead we must decouple the susceptibility at second order by applying a Tyablikov-like decoupling to the equation of motion for the three

point Green's function that appear in the spin susceptibility's equation of motion (8);

$$\begin{aligned} \omega \langle \langle S_i^+ S_k^z; S_j^- \rangle \rangle_\omega &= \langle [S_i^+ S_k^z, S_j^-] \rangle_0 + \langle \langle [S_i^+ S_k^z, H]; S_j^- \rangle \rangle_\omega \\ &= \langle [S_i^+ S_k^z, S_j^-] \rangle_0 + \langle \langle S_i^+ [S_k^z, H] + [S_i^+, H] S_k^z; S_j^- \rangle \rangle_\omega \\ &= (\delta_{i,j} - \delta_{k,j}) \langle S_i^+ S_k^- \rangle_0 \\ &+ \frac{1}{2} \sum_n J_{kn} \langle \langle S_i^+ (S_k^+ S_n^- - S_k^- S_n^+); S_j^- \rangle \rangle_\omega \\ &+ \sum_n J_{in} \langle \langle (S_n^+ S_i^z - S_i^+ S_n^z) S_k^z; S_j^- \rangle \rangle_\omega. \end{aligned} \quad (11)$$

Such a decoupling of the three point Green's function was first introduced by Kondo and Yamaji [41]. There are two types of three operator Green's functions that appear in Eq. 11 these have the form  $\langle \langle S_i^+ S_k^- S_n^+; S_j^- \rangle \rangle$  and  $\langle \langle S_i^+ S_n^z S_k^z; S_j^- \rangle \rangle$ . A Kondo-Yamaji coupling expands these operators in terms finite spin-spin correlations such as  $\langle S_i^+ S_j^- \rangle_0$  and  $\langle S_i^z S_j^z \rangle_0$ ,

$$\begin{aligned} \langle \langle S_i^+ S_k^- S_n^+; S_j^- \rangle \rangle &\rightarrow \alpha \langle S_i^+ S_k^- \rangle_0 \langle \langle S_n^+; S_j^- \rangle \rangle_\omega \\ &+ \alpha \langle S_k^- S_n^+ \rangle_0 \langle \langle S_i^+; S_j^- \rangle \rangle_\omega \end{aligned} \quad (12)$$

and

$$\langle \langle S_i^+ S_n^z S_k^z; S_j^- \rangle \rangle_\omega \rightarrow \alpha \langle S_n^z S_k^z \rangle_0 \langle \langle S_i^+; S_j^- \rangle \rangle_\omega, \quad (13)$$

where  $\alpha$  is found self-consistently. The result can then be Fourier transformed to find the static-susceptibility in Fourier space,  $\chi^{ab}(\mathbf{q}, \omega = 0)$ , as it appears in the RKKY Hamiltonian. This has the form of a matrix equation

$$\sum_b M^{ab}(\mathbf{q}, \omega) \chi^{bc}(\mathbf{q}, \omega) = N^{ac}(\mathbf{q}, \omega), \quad (14)$$

where the matrices  $M^{ab}(\mathbf{q}, \omega)$  and  $N^{ac}(\mathbf{q}, \omega)$  are defined in appendix B, where the full details of the calculation can be found.

The qualitative features of the susceptibility are only weakly dependent on the self-consistent decoupling parameters and result from the Fourier transform of the Heisenberg exchange matrices that make up the matrices in Eq. (14). The eigenvalues of the susceptibility matrix are shown in Fig. 3. Both  $M^{ab}(\mathbf{q}, \omega)$  and  $N^{ac}(\mathbf{q}, \omega)$  have a flat eigenvalue which is a direct consequence of the flat eigenvalue of the Fourier transformed exchange matrix of the nearest neighbor Heisenberg model on kagome. As a consequence the susceptibility  $\chi^{ab}(\mathbf{q}, \omega = 0)$  also has an eigenvalue that is independent of  $\mathbf{q}$ . We can see from Fig. 3 that this is the lowest (largest in magnitude) eigenvalue of the susceptibility matrix.

### IV. MINIMIZATION OF THE RKKY GROUND STATE ENERGY

To minimize the ground state energy we apply an extended version of the Luttinger-Tisza mean field method

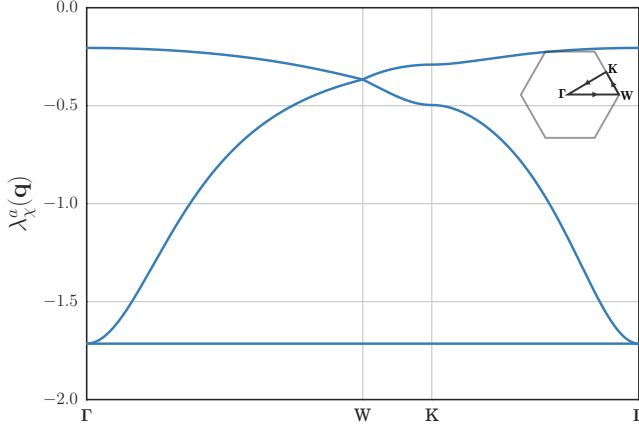


FIG. 3. The eigenvalues of the zero-temperature static susceptibility  $\chi^{ab}(\mathbf{q})$  along a high symmetry path in the Brillouin zone (shown inset). The lowest eigenvalue is constant over the whole Brillouin zone.

[44], which finds the minimum energy configuration of a spin Hamiltonian subject to the classical constraint  $|I_i|^2 = I^2$ . In general this is done by expanding the nuclear spin operators  $\mathbf{I}_a(\mathbf{q})$  on each sublattice  $a$  in terms of the normalized eigenvectors  $U_{a,\nu}(\mathbf{q})$  of the susceptibility  $\chi^{ab}(\mathbf{q})$  where  $a, b = 1, 2, 3$  labels the sublattice component of the vector and  $\nu = 1, 2, 3$  labels the eigenvector. The expansion of the spins then reads,

$$\mathbf{I}_a(\mathbf{q}) = \sum_{\nu} \mathbf{W}_{\nu}(\mathbf{q}) U_{a,\nu}(\mathbf{q}), \quad (15)$$

where the  $\mathbf{W}_{\nu}(\mathbf{q})$  are (orthogonal) directions in spin-space.

A lower bound for the ground state minimum energy is found by relaxing the (classical) constraint  $|I_i|^2 = I^2$  to the constraint  $\sum_i |I_i|^2 = NI^2$ , so that only the average length of the spins in the entire system is equal to  $I$  [45]. Inserting the expansion of the spins in terms of the eigenvectors of the susceptibility, Eq. (15), diagonalizes the susceptibility matrix and we find a lower bound for the ground state energy (see Appendix C for details)

$$E_{\text{GS}} \geq \frac{A^2 I^2 N \lambda_{\min}}{4J}, \quad (16)$$

where  $\lambda_{\min}$  is the minimum eigenvalue of the susceptibility as shown in Fig. (3). For a standard Luttinger-Tisza analysis, based solely on this weaker constraint and in a Bravais lattice, this would specify ordering vectors  $\{\mathbf{Q}_n\}$  where the minimum energy could be found. That is not the case here because the lowest eigenvalue of  $\chi^{ab}(\mathbf{q})$  is the same for all  $\mathbf{q}$  (see Fig. 3) and therefore there is no unique ordering vector or set of ordering vectors  $\{\mathbf{Q}_n\}$  where this lower bound is achieved.

Instead we must search for the lowest energy configurations of nuclear spins by imposing the true constraint on

the length of the nuclear spins at each site [46],  $|I_i|^2 = I^2$ , which in Fourier space reads

$$\frac{1}{N_{Br}} \sum_{\mathbf{k}, \mathbf{G}} \mathbf{I}_{\mathbf{k}}^a \cdot \mathbf{I}_{\mathbf{q}-\mathbf{k}}^a = \sum_{\mathbf{G}} I^2 \delta_{\mathbf{q}\mathbf{G}}, \quad (17)$$

where  $a = 1, 2, 3$  refers to the sublattice and  $\mathbf{G}$  runs over reciprocal lattice vectors.

For a non-Bravais lattice, such as kagome, the hard constraint on the normalization of spins in Eq. (17) can only be fulfilled and simultaneously achieve the energy minimum in Eq. (16) when the eigenvectors of the exchange matrix have the same weight on each site of the unit cell. Otherwise the expansion in terms of eigenvectors in Eq. (15) would require the lengths of spins within a unit cell to vary such that they were inversely proportional to the magnitude of the corresponding component of the susceptibility eigenvector to achieve the minimum energy in Eq. (16) [45].

For the case of the sIRKKY Hamiltonian in Eq. (5), where the lattice of the nuclear spins is the same as the lattice of the electron spins forming the spin liquid, this requires that minima are located at high-symmetry points in the Brillouin zone. The symmetry ensures the diagonals of the susceptibility satisfy the condition

$$\chi^{11}(\mathbf{Q}_n) = \chi^{22}(\mathbf{Q}_n) = \chi^{33}(\mathbf{Q}_n), \quad (18)$$

and the off-diagonals satisfy

$$|\chi^{12}(\mathbf{Q}_n)| = |\chi^{23}(\mathbf{Q}_n)| = |\chi^{13}(\mathbf{Q}_n)|, \quad (19)$$

where  $n$  labels a high-symmetry point in reciprocal space.

The constraints on the susceptibility matrix in Eq. (18) and Eq. (19) mean that the energy minima of Eq. (16) can only be achieved for the nuclear spin system at ordering vectors,  $\mathbf{Q}_n$ , where there is a degeneracy in the eigenvalues of the susceptibility matrix. From Fig. 3 we see that, within the Brillouin zone of the triangular Bravais lattice, this only occurs at the Brillouin zone center, the  $\Gamma$  point, and the Brillouin zone corners, the  $W$  points.

The flat band of the susceptibility leads to each individual high-symmetry point being a minimum of equal energy. Extra exchange interactions in a real material that are not nearest neighbor Heisenberg in nature and (weakly) break the flatness of this band will only have the effect of selecting one of these minima by lowering it with respect to the others. As long as these extra exchange interactions are weak they will not move the minima from the high-symmetry points as these will remain the positions where a true spin expansion as in Eq. (15) can be performed and hence the only positions where the lower bound for the ground state energy in Eq. (16) can be achieved.

## V. ORDERINGS OF THE NUCLEAR MOMENTS AT ENERGY MINIMA

To obtain the orderings that correspond to the energy minima found in Sec. IV at the high symmetry points we



must analyse the eigenvectors of the susceptibility matrix at the ordering vector  $\mathbf{Q}_n$ , the eigenvectors tell us the relative angle between the three spins of the unit cell. As a result of considering individual sites of the unit cell the orderings differ between the high-symmetry points of the extended Brillouin zone. This extension is required because the Brillouin zone of the underlying triangular lattice does not include the complete information for structures which take into account individual sites of the kagome lattice.

Fig. 3 shows that two separate scenarios can occur at the high symmetry points: either the lowest eigenvalue of susceptibility is not degenerate, as found at the corners of the first Brillouin zone, or it is two-fold degenerate, as found at the  $\Gamma$  point.

For the minima with a singular lowest eigenvalue the expansion of the nuclear spin in terms of the eigenvectors of the susceptibility, Eq. (15), includes only one vector in spin-space. As such at non-degenerate high symmetry points all spins within the unit cell must be either parallel or anti-parallel. Only two orders are possible depending on the sign of susceptibility matrix elements at these points: either the spins create a two up–one down order with one spin in the unit cell anti-parallel to the other two, or they are all parallel. The former is found at the corners of the first Brillouin zone, whereas the latter is found at the corners of the extended Brillouin zone, these are shown in Fig. 4 as  $\square$ s and  $\triangle$ s respectively.

For the high symmetry points where the lowest eigenvalue of the susceptibility is degenerate the expansion of the spins in Eq. (15) allows for two vectors in spin space. As such the nuclear spins within a unit cell can now lie at angles to each other in the same plane, with the angle given by the sign of the components of the eigenvectors of the susceptibility matrix. From the constraints on the susceptibility matrix in Eq. (18) and Eq. (19) the only possible orderings are all spins rotated  $2\pi/3$  relative to each other, or a central spin with the other two spins rotated  $\pm\pi/3$  to this central spin. The former is found at the  $\Gamma$ -point and the latter at the center of the edges of the extended Brillouin zone, these are shown in Fig. 4 as the  $\circ$  and  $\diamond$ s respectively.

To translate the orderings back into a real space ordering we find the planes where  $\mathbf{Q}_n \cdot \mathbf{r}_m = 2\pi m$  for the sites of the underlying triangular lattice of the unit cell. The unit cells on these planes have each nuclear spin on equivalent sites aligned and the sites of the unit cells in any intermediate planes have their spins on each site rotated equally. Example orderings are shown in Fig. 5. It should be noted that the spins are not constrained to the same plane as the kagome lattice because only the relative angle between spins is important to achieve an energy minimum.

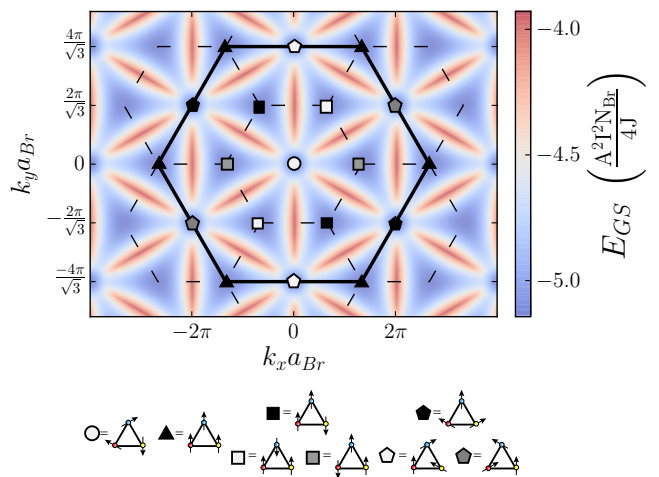


FIG. 4. Density plot of minimum energy in k-space calculated numerically by enforcing the hard spin length constraint of Eq. (17). The positions of ground state energies that achieve the lower bound of Eq. (16) within the extended Brillouin zone (solid black line) are highlighted. The orderings at these points are shown in the key below.

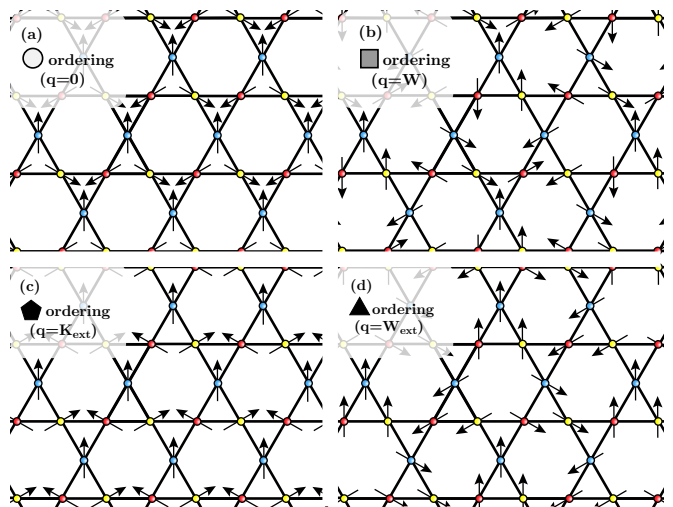


FIG. 5. The four distinct types of orderings shown in real space. The corresponding symbols from Fig. 4 and the position within the Brillouin zone are shown in the upper right corner of each pane.

## VI. STABILITY OF THE NUCLEAR MOMENT ORDERINGS ON KAGOME

To test the stability of the mean field orderings found in Sec. V under fluctuations we investigate how spin-waves modify or destroy those orderings.

To do this we select a basis for the directions in spin-space such that the spin on the first site of the unit cell is parallel to the z-axis,  $\mathbf{I}_1(\mathbf{Q}_n) \parallel \hat{\mathbf{z}}$ . Within this basis, because the orderings are planar, we can effec-

tively rotate the remaining spins in the unit cell about the y-axis to be parallel with the first spin by defining  $\mathbf{I}'_2(\mathbf{Q}_n) = R(\theta_2)\mathbf{I}_2(\mathbf{Q}_n)$  and  $\mathbf{I}'_3(\mathbf{Q}_n) = R(\theta_3)\mathbf{I}_3(\mathbf{Q}_n)$ , where  $R(\theta_i)$  is the rotation matrix about y-axis and  $\theta_i$  is the angle between the first spin and the spin on site  $i$ . Inserting this into Eq. (5) gives,

$$H = \frac{A^2}{4J} \sum_{\mathbf{q}, a, b} \chi^{ab}(\mathbf{Q}_n, \omega = 0) \left\{ \begin{aligned} & \cos(\theta_a - \theta_b) (I'_a{}^z(\mathbf{Q}_n) I'_b{}^z(-\mathbf{Q}_n) + I'_a{}^x(\mathbf{Q}_n) I'_b{}^x(-\mathbf{Q}_n)) \\ & + \sin(\theta_a - \theta_b) (I'_a{}^x(\mathbf{Q}_n) I'_b{}^z(-\mathbf{Q}_n) - I'_a{}^z(\mathbf{Q}_n) I'_b{}^x(-\mathbf{Q}_n)) \\ & + I'_a{}^y(\mathbf{Q}_n) I'_b{}^y(-\mathbf{Q}_n) \end{aligned} \right\}. \quad (20)$$

Since the nuclear spins are large we can take a  $1/I$  Holstein-Primakoff expansion [47], which in Fourier space reads

$$\begin{aligned} I'_{a, \mathbf{Q}_n}{}^z(\mathbf{q}) &= N_{Br} \delta_{\mathbf{q}0} I - \sum_{\mathbf{p}} a_{\mathbf{p}}^{\dagger a} a_{\mathbf{p}+\mathbf{q}}^a, \\ I'_{a, \mathbf{Q}_n}{}^x(\mathbf{q}) &= \sqrt{\frac{N_{Br} I}{2}} (a_{\mathbf{q}}^a + a_{-\mathbf{q}}^{\dagger a}), \\ I'_{a, \mathbf{Q}_n}{}^y(\mathbf{q}) &= i \sqrt{\frac{N_{Br} I}{2}} (a_{-\mathbf{q}}^{\dagger a} - a_{\mathbf{q}}^a), \end{aligned} \quad (21)$$

where  $a_{\mathbf{q}}^{\dagger a}, a_{\mathbf{q}}^a$  are the bosonic creation and annihilation operators, respectively, for magnons at momentum  $\mathbf{q}$  on sublattice  $a$  about the mean field equilibrium ordering vector  $\mathbf{Q}_n$ .

Inserting the Holstein-Primakoff expansion in Eq. (21) to zeroth-order in bosonic operators,  $\mathcal{O}(I^2)$ , into the basis-rotated Hamiltonian Eq. (20) gives the minimal ground state energy

$$\begin{aligned} H^{(0)} &= \frac{A^2}{4J} \sum_{\mathbf{q}, a, b} \chi^{ab}(\mathbf{Q}_n, 0) \cos(\theta_a - \theta_b) N_{Br} I^2 \\ &= \frac{A^2 N I^2 \lambda_{\min}}{4J}, \end{aligned} \quad (22)$$

where the second line follows from the fact that we chose the angles  $\theta_a$  for our equilibrium ordering such that the matrix  $I^2 \cos(\theta_a - \theta_b) = \mathbf{I}_a(\mathbf{Q}_n) \cdot \mathbf{I}_b(-\mathbf{Q}_n)$  outputs the minimum eigenvalue of  $\chi^{ab}(\mathbf{Q}_n, 0)$ .

The next order of the expansion comes from mixed  $x$  and  $z$  spin components in Eq. (20). These will equal zero if

$$\sum_b \chi^{ab}(\mathbf{Q}_n, 0) \sin(\theta_a - \theta_b) = 0. \quad (23)$$

The construction of  $\cos(\theta_a - \theta_b)$  to minimize the energy of the sLRKKY Hamiltonian in Eq. (5) means that Eq. (23) must hold or else one could rotate and achieve a lower minimum than that set by  $\theta_a$ . Conversely the condition in Eq. (19) - that the exchange interaction has equal magnitude on each site - and the above condition in Eq. (23) - that the effective magnetic field on

each site vanishes - set limits on the possible orderings we could find. Given these two conditions the mutual angles between the three spins in the unit cell must either be collinear, which automatically satisfies Eq. (23), or have angles of  $60^\circ$  or  $120^\circ$  between them such that the terms from the susceptibility matrix cancel. As such the only orderings that can satisfy both conditions are those orderings already found in Sec. V.

The second order terms,  $\mathcal{O}(I)$ , provide the excitation spectrum of the spin-waves, these can be written in as a block  $6 \times 6$  Bogoliubov matrix

$$H^{(2)} = \frac{A^2 N_{Br} I}{8J} \sum_{\mathbf{q}} (\mathbf{a}_{-\mathbf{q}}^\dagger, \mathbf{a}_{\mathbf{q}}) \begin{pmatrix} \mathbf{\Gamma} & \mathbf{\Lambda} \\ \mathbf{\Lambda} & \mathbf{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{-\mathbf{q}} \\ \mathbf{a}_{\mathbf{q}}^\dagger \end{pmatrix}, \quad (24)$$

where we have defined the vectors of bosonic operators on sublattice  $a = 1, 2, 3$

$$[\mathbf{a}_{\mathbf{q}}^\dagger]^a = a_{\mathbf{q}}^{a\dagger}, [\mathbf{a}_{\mathbf{q}}]^a = a_{\mathbf{q}}^a, \quad (25)$$

and similarly the  $3 \times 3$  matrices  $\mathbf{\Gamma}$  and  $\mathbf{\Lambda}$  with entries

$$\begin{aligned} [\mathbf{\Gamma}]_{ab} &= 2\delta_{ab} \sum_c \cos(\theta_a - \theta_b) \chi_{ac}(\mathbf{Q}_n) \\ &\quad - \cos(\theta_a - \theta_b) \chi_{ab}(\mathbf{Q}_n + \mathbf{q}) - \chi_{ab}(\mathbf{Q}_n + \mathbf{q}), \end{aligned} \quad (26)$$

and

$$[\mathbf{\Lambda}]_{ab} = (-\cos(\theta_a - \theta_b) + 1) \chi_{ab}(\mathbf{Q}_n + \mathbf{q}). \quad (27)$$

The Hamiltonian is diagonalized via a Bogoliubov transformation for 3-species of bosons to find the spin-wave spectra  $\omega_a(\mathbf{q})$ , with  $a = 1, 2, 3$ . The diagonalization is presented in Appendix D. An example spectra for the  $\mathbf{Q} = 0$  ordering is shown in Fig. 6.

Remarkably we find that for each ordering there is a flat spin-wave band at strictly zero energy for all momenta  $\mathbf{q}$  of reciprocal space. As a result of this zero-energy band the orderings found in Sec. V will evaporate at any finite temperature. A proof of the existence of a zero-energy eigenvalue can be found in Appendix E.

The existence of the zero-energy band in the linear spin-wave theory is a direct consequence of the flat band of the kagome susceptibility. This is can be seen from the fact that only the hard constraint on spin length produced the energy minimum and such a constraint can only be enforced by terms beyond linear spin-waves. In other words the fluctuations to linear order reduce the length of spin in such a way that it is unaware of the hard constraint on spin length and because this is the origin of the ordering minimum it is therefore unsurprising that to linear order we find a flat zero-energy band. Higher orders of the spin-wave expansion or exchange interactions that are not nearest neighbor Heisenberg can break the flatness of this band, however the curvature will remain very small. As such the corresponding ordering temperature in the thermodynamic limit will remain zero because a  $\sim q^2$  dispersion still leads to a divergence in magnon occupation for 2D systems [29, 30]. In a finite

system with these interactions present the temperature will be finite but very low due to the small amount of curvature in the band.

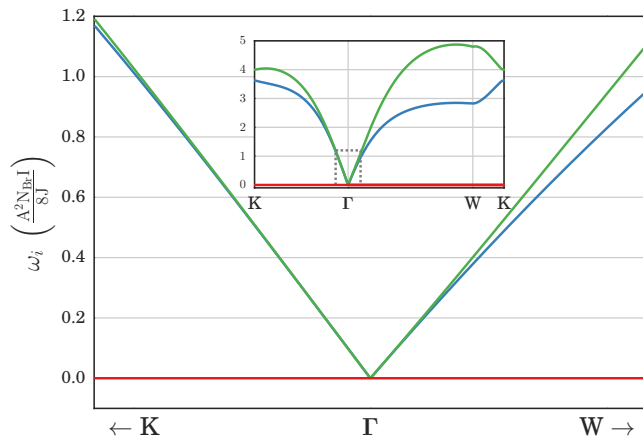


FIG. 6. The three spin-wave spectra for the ordering  $\bigcirc$  centred about the  $\Gamma$ -point. The full spectrum along the same high-symmetry path as Fig. 3 is shown inset. A zero-energy flat band is found at all  $\mathbf{q}$  of reciprocal space.

## VII. STABILIZATION OF NUCLEAR ORDER WITH A MAGNETIC FIELD

Whilst the flatness of the magnon band makes order impossible at any finite temperature, it is possible to use a magnetic field to select a ground state order from those found in Fig. 4 and create a gap in the associated spin wave spectrum which has the effect of stabilising the selected order against temperature.

To show this we assume a uniform magnetic field across the whole lattice which in Fourier space induces a Zeeman term  $\sum_a B I_a^z(\mathbf{Q} = \mathbf{0} + \mathbf{G})$ , where  $\mathbf{G}$  are the reciprocal lattice vectors of the triangular Bravais lattice. Such a term prefers orderings of the nuclear spins that have a net magnetic moment and will seek to align the net moment with the field and lower the ground state energy.

The only set of orderings which have a net moment are those found at the 3 different  $\bigcirc$  orderings in the extended Brillouin zone (see Figs. 4 and 5), here the optimal arrangement is when the central spin aligns with the field and so the total ground state energy – including the two other spins at  $60^\circ$  to the central spin – is reduced to  $E_{GS}^0 - 2g_{\text{nuc}}INB$ .

The spin-wave analysis then proceeds as before in Sec. VI by taking a Holstein-Primakoff expansion about these ordering vector in momentum space. Unlike in the free field case the linear terms no longer cancel because the additional Zeeman term contains components of  $I_2^x$ ,  $I_3^x$  from the spins that are not fully aligned with the field.

This causes linear terms in the spin-wave operators,

$$H^{(1)} = \frac{\sqrt{3}g_{\text{nuc}}N_{Br}IB}{4}(-a_{\mathbf{q}}^2 - a_{\mathbf{q}}^2 + a_{\mathbf{q}}^3 + a_{\mathbf{q}}^3). \quad (28)$$

These terms represent the fact that the effective magnetic field from the RKKY and the external field now compete and the nuclear spins will cant upwards slightly from the true  $60^\circ$  from the central spin found at zero-field, further increasing magnetic field would eventually form a ferromagnetic arrangement. For our purposes we assume that fields are sufficiently small that the order can be approximated by the original mean field RKKY result. Additionally these small magnetic fields will have little effect on the qualitative features of the spin liquid since the Zeeman energy of the electron spins will be much smaller than the exchange Heisenberg energy  $J$ .

The second order terms of the Holstein-Primakoff expansion, which determine the magnon spectrum, will now be of the form

$$H^{(2)} = H_{RKKY}^{(2)} + g_{\text{nuc}}BN_{Br}I \sum_{a,\mathbf{q}} a_{\mathbf{q}}^{\dagger a} a_{\mathbf{q}}^a, \quad (29)$$

where  $H_{RKKY}^{(2)}$  is the zero-field RKKY term at quadratic order. The result is that a constant is added to the terms diagonal in spin-wave operators and this opens a gap of the order of the order of  $BN_{Br}I$  to the spin-wave spectra – including the flat band which previously destabilized the ordering at zero-field. As a consequence the ordering is stable until the temperature is of the order of the gap, at which point the lowest band in the spin-wave spectra is quickly occupied and the order will be destroyed. Using DQVOF as an example the nuclear g-factor of vanadium [39] is  $g_{\text{nuc}} = 1.47$  and so requires a field  $B \sim 500\text{mT}$  per mK. For small fields this would correspond loosely to the limit of current state of the art experiments.

## VIII. DISCUSSION

We have shown that a spin liquid can mediate an RKKY-type interaction between localized moments such as nuclear spins. This interaction has a distinct character to the standard RKKY interaction of fermionic systems; for example, the absence of a  $k_F$  means that the minimum wavelength for destabilizing fluctuations is set by the Brillouin zone boundaries.

Our working example, the kagome anti-ferromagnet, illustrates that such an interaction can induce distinct and varied orderings of the nuclear moments. Interestingly, due to a zero-energy flat magnon band, these orderings destabilize at any finite temperature regardless of if the system is infinite or finite in size. Despite this we also show that a small magnetic field can stabilise such an ordering to within potentially experimentally achievable temperatures without altering the underlying physics.



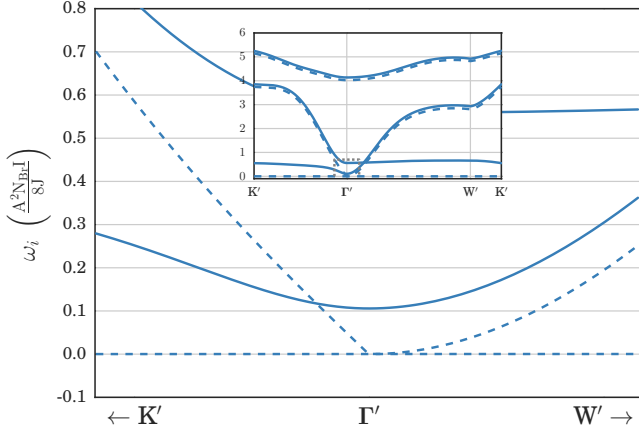


FIG. 7. The spin-wave spectra at zero magnetic field (red, dashed) and finite field parallel to the central spin (blue) for the  $\diamond$  orderings centered about the center of the edges of the extended Brillouin zone. The flat band of spin waves at no magnetic field is gapped out at finite field. The full spectrum along the same high-symmetry path as Fig. 3 is shown inset.

This means, as a result of recent advances in developing spin liquid materials, such an interaction could have directly testable experimental consequences with today's state of the art.

Our treatment throughout this work has been general and we expect such an analysis should also be applicable to other spin liquids such as the 3D hyperkagome lattice and the Kitaev model realised on 2D and 3D lattices.

## ACKNOWLEDGMENTS

The authors would like to thank A. Rosch, C.A. Hooley, and K. O'Brien for helpful discussions. H.F.L. thanks the DFG under CRC 1238 and Bonn-Cologne Graduate School of Physics and Astronomy (BCGS) for their financial support.

## Appendix A: Formulation of effective slRKKY Hamiltonian

Since the Hyperfine coupling,  $A$ , is a small energy scale we can perform a Schrieffer-Wolff transformation, eliminating terms linear in  $A$  and keeping terms up to order  $A^2$ . After integrating out the electron spin degrees of freedom in Eq. 1 we obtain an effective interaction Hamiltonian of the form [30]

$$H_{\text{SW}} = H_{\text{sl}} + \frac{1}{2}[S, [S, H_{\text{sl}}]], \quad (\text{A1})$$

with  $S$  defined by  $H_{\text{hyp}} + [S, H_{\text{sl}}] = 0$ . This is solved such that

$$\frac{1}{2}[S, [S, H_{\text{sl}}]] = -\frac{i}{2} \int_0^\infty dt e^{-\eta t} [H_{\text{hyp}}(t), H_{\text{hyp}}(0)], \quad (\text{A2})$$

where the time evolution of the hyperfine Hamiltonian is given by  $H_{\text{hyp}} = e^{iH_{\text{sl}}t} H_{\text{hyp}}(0) e^{-iH_{\text{sl}}t}$  and  $\eta \rightarrow 0^+$  ensures convergence of the integral. Inserting the definition of  $H_{\text{hyp}}$  gives

$$\begin{aligned} \frac{1}{2}[S, [S, H_{\text{sl}}]] &= -\frac{iA^2}{2} \sum_{i,j} \int_0^\infty dt e^{-\eta t} [\mathbf{I}_i \cdot \mathbf{S}_i(t), \mathbf{I}_j \cdot \mathbf{S}_j(0)] \\ &= -\frac{iA^2}{2} \sum_{i,j} \int_0^\infty dt e^{-\eta t} \left\{ I_i^\alpha I_j^\beta [S_i^\alpha(t), S_j^\beta] \right. \\ &\quad \left. + [I_i^\alpha, I_j^\beta] S_j^\beta S_i^\alpha(t) \right\}, \end{aligned} \quad (\text{A3})$$

where summation over Greek indices is implied. Integrating out the electron spin degrees of freedom from  $H_0$  leaves us with the effective RKKY Hamiltonian:

$$\begin{aligned} H_{\text{eff}} &= -\frac{iA^2}{2} \sum_{i,j} \int_0^\infty dt e^{-\eta t} I_i^\alpha I_j^\beta \langle [S_i^\alpha(t), S_j^\beta] \rangle_0 \\ &= \frac{A^2}{8J} \sum_{i,j} \chi_{ij}^{\alpha\beta}(\omega=0) I_i^\alpha I_j^\beta, \end{aligned} \quad (\text{A4})$$

where  $\alpha, \beta = x, y, z$ ;  $\chi_{ij}^{\alpha\beta}$  is the real space static spin susceptibility; and the second term of the last line of Eq. (A3) is zero due to the spin liquid being a non-magnetic phase. In the non-magnetic spin liquid state the susceptibility is independent of the directions in spin-space [38] and hence we can write  $2\chi_{ij}^{\alpha\beta} = \chi_{ij}^{+-} = \chi_{ij}$ .

In the presence of a small magnetic field, as in Sec. VII, the spin liquid itself may develop a magnetization that will scale as  $B/J$ . From Eq. (A3), the resulting effective Zeeman field from this magnetization acting on the nuclear spins scales as  $A^2 B/J^2$  and this will be negligibly small compared to the bare Zeeman energy of the nuclear spins.

We define the Fourier transform of the susceptibility with respect to the underlying Bravais lattice [38] such that

$$\chi^{ab}(\mathbf{q}, \omega) = \frac{1}{N_{Br}} \sum_{i,j} \chi_{ij}^{ab} e^{-i\mathbf{q} \cdot (\mathbf{R}_i^a - \mathbf{R}_j^b)}, \quad (\text{A5})$$

where  $N_{Br}$  is the number of sites on the underlying triangular Bravais lattice;  $a, b$  refer to the sites in the unit cell as shown in Fig. 2; and  $\mathbf{R}_i^a = \mathbf{R}_i + \mathbf{T}_a$ , with  $\mathbf{R}_i$  the lattice vector on the triangular Bravais lattice and  $\mathbf{T}_a$  the vector from the center of a lattice triangle to the site  $a$  of the unit cell.

Inserting the definition of the Fourier transformed susceptibility into Eq. (A4) we obtain the effective RKKY Hamiltonian between nuclear spins in Fourier space,

$$H_{\text{eff}} = \frac{A^2}{2J} \sum_{\mathbf{q}, a, b} \chi^{ab}(\mathbf{q}, \omega=0) \mathbf{I}_{\mathbf{q}}^a \cdot \mathbf{I}_{-\mathbf{q}}^b, \quad (\text{A6})$$

as defined in (5).

### Appendix B: Kondo-Yamaji decoupling of electron spin susceptibility

We perform the Kondo-Yamaji decoupling in the same manner as calculated in previous work on the KAFM [38, 42]. Using the fact that in the non-magnetic spin liquid phase  $\langle S_i^z \rangle = 0$  we can write the first order expansion of the susceptibility, Eq. (8), as

$$\begin{aligned} \omega \chi_{ij}(\omega) &= \omega \langle \langle S_k^+; S_j^- \rangle \rangle_\omega \\ &= \sum_k J_{ik} [\langle \langle S_i^z S_k^+; S_j^- \rangle \rangle_\omega - \langle \langle S_i^+ S_k^z; S_j^- \rangle \rangle_\omega]. \end{aligned} \quad (\text{B1})$$

We then insert the decoupled equation of motion for the three-point functions  $\langle \langle S_i^z S_k^+; S_j^- \rangle \rangle_\omega$  and  $\langle \langle S_i^+ S_k^z; S_j^- \rangle \rangle_\omega$  from Eq. (11) with the Kondo-Yamaji decoupling parameters from Eqs. (12) and (13). This gives an equation entirely in terms of  $\langle \langle S_k^+; S_j^- \rangle \rangle_\omega$ , self-consistent parameters, and exchange matrices of the Heisenberg model such that

$$\begin{aligned} \omega^2 \langle \langle S_k^+; S_j^- \rangle \rangle_\omega &= 2c_1 J_{ij} - 8J \delta_{ij} c_1 \\ &\quad - (c + 2\alpha c_1) J \sum_k J_{ik} \langle \langle S_k^+; S_j^- \rangle \rangle_\omega \\ &\quad + 4cJ^2 \langle \langle S_k^+; S_j^- \rangle \rangle_\omega \\ &\quad + \alpha c_1 J \sum_k J'_{ik} \langle \langle S_k^+; S_j^- \rangle \rangle_\omega \end{aligned} \quad (\text{B2})$$

where

$$\langle S_i^+ S_j^- \rangle = \begin{cases} c_1 & i, j \text{ nearest neighbors (NN)} \\ c_2 & i, j \text{ next-NN (shortest distance)} \\ c'_2 & i, j \text{ next-NN (longest distance),} \end{cases} \quad (\text{B3})$$

$\alpha$  is the self-consistent parameter of the Kondo-Yamaji decoupling, and  $J'_{ij}$  is defined by

$$J'_{ij} = \begin{cases} J > 0, & \text{if } i \text{ and } j \text{ are next-nearest neighbors} \\ 0, & \text{otherwise.} \end{cases} \quad (\text{B4})$$

Replacing  $\chi_{ij}(\omega) = \langle \langle S_k^+; S_j^- \rangle \rangle_\omega$  and taking the Fourier transform with respect to the triangular Bravais lattice gives a matrix equation of the form

$$\sum_b M^{ab}(\mathbf{q}, \omega) \chi^{bc}(\mathbf{q}, \omega) = N^{ac}(\mathbf{q}, \omega), \quad (\text{B5})$$

where we have defined the matrices

$$\begin{aligned} M^{ab}(\mathbf{q}, \omega) &= (\omega^2 - 4cJ^2) \delta_{ab} \\ &\quad + J(c + 2\alpha c_1) J_{ab}(\mathbf{q}) - \alpha c_1 J'_{ab}(\mathbf{q}) \end{aligned} \quad (\text{B6})$$

and

$$N^{ab}(\mathbf{q}, \omega) = 2c_1 J_{ab}(\mathbf{q}) - 4J c_1 \delta_{ab}, \quad (\text{B7})$$

with  $J_{ab}(\mathbf{q})$  and  $J'_{ab}(\mathbf{q})$  the Fourier transforms of the exchange matrices  $J_{ij}$  and  $J'_{ij}$  of the Heisenberg model on kagome [38] such that

$$J_{ab}(\mathbf{q}) = 2J \cos(\mathbf{q} \cdot \mathbf{r}_{ab}) - \delta_{ab}, \quad (\text{B8})$$

where  $\mathbf{r}_{ab}$  are the real space lattice vectors between neighboring sites  $a$  and  $b$ , and

$$J'_{ab}(\mathbf{q}) = J \sum_{\mathbf{r}'_{ab}} \cos(\mathbf{q} \cdot \mathbf{r}'_{ab}), \quad (\text{B9})$$

where  $\mathbf{r}'_{ab}$  are the four real space lattice vectors between next-nearest-neighboring sites (by lattice distance)  $a$  and  $b$ . Due to the frustrated nature of the kagome lattice both  $J_{ab}(\mathbf{q})$  and  $J'_{ab}(\mathbf{q})$  have an eigenvalue which is independent of  $\mathbf{q}$  and, as a direct consequence of only these two matrices appearing in Eq. (B5), the susceptibility matrix  $\chi^{bc}(\mathbf{q}, \omega = 0)$  also has a flat band.

The matrix equation Eq. (B5) can then be solved to find the static-susceptibility matrix  $\chi^{bc}(\mathbf{q}, \omega = 0)$ . The eigenvalues of this matrix are shown in Fig. 3 and the qualitative features only weakly depend on the values of the decoupling parameters  $c_1$ ,  $c_2$ , and  $c'_2$ , with a lowest eigenvalue independent of  $\mathbf{q}$ .

### Appendix C: Ground state energy lower bound

Inserting the expansion of the spins in terms of the eigenvectors of the susceptibility matrix  $\chi^{ab}(\mathbf{q})$  from Eq. (15) into the slRKKY Hamiltonian of Eq. (5) reads

$$H_{\text{eff}} = \frac{A^2}{4J} \sum_{\mathbf{q}, a, b, \nu, \nu'} \chi^{ab}(\mathbf{q}, \omega = 0) U_{\mathbf{q}}^{a, \nu} U_{-\mathbf{q}}^{b, \nu'} \mathbf{W}^\nu \cdot \mathbf{W}^{\nu'}. \quad (\text{C1})$$

Using the fact that  $U_{\mathbf{q}}^{a, \nu}$  is an eigenvector of the static susceptibility matrix with eigenvalue  $\lambda^\nu(\mathbf{q})$  gives

$$H_{\text{eff}} = \frac{A^2}{4J} \sum_{\mathbf{q}, b, \nu, \nu'} \lambda^\nu(\mathbf{q}) U_{\mathbf{q}}^{b, \nu} U_{-\mathbf{q}}^{b, \nu'} \mathbf{W}^\nu \cdot \mathbf{W}^{\nu'}. \quad (\text{C2})$$

Because the eigenvectors  $U_{\mathbf{q}}^{a, \nu}$  are orthonormal this then reduces to

$$H_{\text{eff}} = \frac{A^2}{4J} \sum_{\mathbf{q}, \nu} \lambda^\nu(\mathbf{q}) \mathbf{W}^\nu \cdot \mathbf{W}^\nu. \quad (\text{C3})$$

The weak constraint on average spin length in Fourier space reads  $\sum_{\mathbf{q}, a} I_{\mathbf{q}}^a \cdot I_{-\mathbf{q}}^a = NI^2$  means that from the expansion Eq. (15) we require  $\mathbf{W}^\nu \cdot \mathbf{W}^\nu = I^2$  and so the minimum ground state energy with this weak constraint applied is

$$E_{\text{GS}} = \frac{A^2 NI^2 \lambda_{\min}}{4J}. \quad (\text{C4})$$

## Appendix D: Spin-wave diagonalization

We wish to solve the general problem of finding the transformation of  $N$  species of bosons that diagonalizes a block-diagonal spin-wave Hamiltonian as in Eq. (24) in terms of new operators  $\mathbf{d}(\mathbf{q})$  and  $\mathbf{d}^\dagger(\mathbf{q})$  to a Hamiltonian of the form

$$\begin{aligned} H &= \sum_{\mathbf{q}} (\mathbf{a}_{-\mathbf{q}}^\dagger, \mathbf{a}_{\mathbf{q}}) \begin{pmatrix} \mathbf{\Gamma} & \mathbf{\Lambda} \\ \mathbf{\Lambda} & \mathbf{\Gamma} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{-\mathbf{q}} \\ \mathbf{a}_{\mathbf{q}}^\dagger \end{pmatrix} \\ &= \sum_{\mathbf{q}} (\mathbf{d}_{-\mathbf{q}}^\dagger, \mathbf{d}_{\mathbf{q}}) \begin{pmatrix} \mathbf{\Delta}(\mathbf{q}) & 0 \\ 0 & \mathbf{\Delta}(\mathbf{q}) \end{pmatrix} \begin{pmatrix} \mathbf{d}_{-\mathbf{q}} \\ \mathbf{d}_{\mathbf{q}}^\dagger \end{pmatrix}, \end{aligned} \quad (\text{D1})$$

where  $[\mathbf{\Delta}]_{ab} = \omega_a \delta_{ab}$  is a diagonal matrix of the spin wave spectra and the transformation matrix  $\mathbf{D}$  between operators

$$\begin{pmatrix} \mathbf{d}_{-\mathbf{q}} \\ \mathbf{d}_{\mathbf{q}}^\dagger \end{pmatrix} = \mathbf{D} \begin{pmatrix} \mathbf{a}_{-\mathbf{q}} \\ \mathbf{a}_{\mathbf{q}}^\dagger \end{pmatrix} = \begin{pmatrix} \mathbf{u} & \mathbf{v} \\ \mathbf{v} & \mathbf{u} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{-\mathbf{q}} \\ \mathbf{a}_{\mathbf{q}}^\dagger \end{pmatrix}, \quad (\text{D2})$$

with  $N \times N$  matrices  $\mathbf{u}$  and  $\mathbf{v}$ . To preserve bosonic commutation relations these must satisfy

$$\mathbf{u}\mathbf{u}^T - \mathbf{v}\mathbf{v}^T = \mathbb{1}_N \quad (\text{D3})$$

and

$$\mathbf{u}\mathbf{v}^T = \mathbf{v}\mathbf{u}^T, \quad (\text{D4})$$

where  $\mathbb{1}_N$  is the  $N \times N$  identity matrix. Eq. (D3) and (D4) require that

$$\mathbf{D}^{-1} = \begin{pmatrix} \mathbf{u} & -\mathbf{v} \\ -\mathbf{v} & \mathbf{u} \end{pmatrix}, \quad (\text{D5})$$

which enables us to write the diagonalization problem as an eigenvalue problem of the form

$$\begin{pmatrix} \mathbf{\Gamma} & -\mathbf{\Lambda} \\ \mathbf{\Lambda} & -\mathbf{\Gamma} \end{pmatrix} = \mathbf{D} \begin{pmatrix} \mathbf{\Delta}(\mathbf{q}) & 0 \\ 0 & -\mathbf{\Delta}(\mathbf{q}) \end{pmatrix} \mathbf{D}^{-1}, \quad (\text{D6})$$

which means that to find the spin-wave spectra contained within  $\mathbf{\Delta}$  we must find the eigenvalues of the left hand side of Eq. (D6).

## Appendix E: Zero-energy spin-wave band

To see that the the spin-waves of the spin liquid RKKY interaction on kagome have a zero-energy flat band for all  $\mathbf{q}$  we use the fact that the determinant of the matrix in Eq. (D6) is the product of the eigenvalues and so the matrix must have zero determinant - and hence be non-invertible - if it has a zero eigenvalue. Since a zero determinant is unchanged by addition of rows and columns; subtracting the second row from the first and then subtracting the first column from the second. Leads to,

$$\begin{aligned} \begin{pmatrix} \mathbf{\Gamma} & -\mathbf{\Lambda} \\ \mathbf{\Lambda} & -\mathbf{\Gamma} \end{pmatrix} &\rightarrow \begin{pmatrix} \mathbf{\Gamma} - \mathbf{\Lambda} & -\mathbf{\Lambda} - \mathbf{\Gamma} \\ \mathbf{\Lambda} & -\mathbf{\Gamma} \end{pmatrix} \\ &\rightarrow \begin{pmatrix} \mathbf{\Gamma} - \mathbf{\Lambda} & 0 \\ \mathbf{\Lambda} & -\mathbf{\Gamma} - \mathbf{\Lambda} \end{pmatrix}, \end{aligned} \quad (\text{E1})$$

which is now in the form of a block triangular matrix. We can therefore write the determinant as  $\Delta = \det\{\mathbf{\Gamma} - \mathbf{\Lambda}\} \det\{-\mathbf{\Gamma} - \mathbf{\Lambda}\}$ . From the definitions Eq. (26) & Eq. (27) for  $\mathbf{\Gamma}$  &  $\mathbf{\Lambda}$  the first block matrix in terms of the susceptibility reads

$$\begin{aligned} [\mathbf{\Gamma}]_{ab} - [\mathbf{\Lambda}]_{ab} &= 2\delta_{ab} \sum_c \cos(\theta_a - \theta_c) \chi_{ac}(\mathbf{Q}_n) \\ &\quad - 2\chi_{ab}(\mathbf{Q}_n + \mathbf{q}) \\ &= 2\lambda_{\min} \delta_{ab} - 2\chi_{ab}(\mathbf{Q}_n + \mathbf{q}), \end{aligned} \quad (\text{E2})$$

where the  $\lambda_{\min}$  in the final line follows from the choice of the spin orientations  $\theta_i$  such that we achieve the mean field minimum energy. The matrix in Eq. (E2) therefore has a zero-eigenvalue from the cancellation of the diagonal of the first term and the  $\lambda_{\min}$  flat band of the susceptibility,  $\chi^{ab}(\mathbf{Q}_n + \mathbf{q})$ . From this it follows that  $\det\{\mathbf{\Gamma} - \mathbf{\Lambda}\} = 0$  and hence the original Bogoliubov matrix of spin-wave excitations in Eq. (E1) as has a zero-energy spectrum for all  $\mathbf{q}$ .

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